Two inequalities
https://www.linkedin.com/groups/8313943/8313943-6361509260144312323
Let $x,y,z,t > 0$, prove that
(**a**) $x+y+z+3/(1/x+1/y+1/z) \geq 4(xyz)^{1/3}$.
(**b**) $x+y+z+t+4/(1/x+1/y+1/z+1/t) \geq 5(xyzt)^{1/4}$.
Solution by Arkady Alt, **San Jose**,**California**, **USA**.
(**a**) Since $(x+y+z)^2 \geq 3(xy+yz+zx)$ we have
(**1**) $3/(1/x+1/y+1/z) = \dfrac{3xyz}{xy+yz+zx} \geq \dfrac{9xyz}{(x+y+z)^2}$. Thus, suffice to prove inequality
(**2**) $x+y+z+\dfrac{9xyz}{(x+y+z)^2} \geq 4(xyz)^{1/3}$.
Let $x+y+z=1$(due homogeneity of (**2**)) and let $u := (xyz)^{1/3} \leq \dfrac{x+y+z}{3} = \dfrac{1}{3}$.
Then inequality (**2**) becomes $1+9u^3 \geq 4u$ and we have
$1+9u^3-4u = (1-3u)(1-u-3u^2) \geq (1-3u)\left(1-\frac{1}{3}-3\cdot\frac{1}{3^2}\right) = \dfrac{1-3u}{3} \geq 0.$
Remark.
Denoting $A := \dfrac{x+y+z}{3}, H := \left(\dfrac{x^{-1}+y^{-1}+z^{-1}}{3}\right)^{-1}$ and $G := (xyz)^{1/3}$ we can rewrite inequality (**a**) as $3A+H \geq 4G$ and inequality (**1**) as $A^2H \geq G^3$.
(**b**) Since by Maclaurin's inequality $\dfrac{x+y+z+t}{4} \geq \left(\dfrac{xyz+xyt+xzt+yzt}{4}\right)^3 \iff$ $(x+y+z+t)^3 \geq 16(xyz+xyt+xzt+yzt)$ we have
(**3**) $4/(1/x+1/y+1/z+1/t) = \dfrac{4xyzt}{xyz+xyt+xzt+yzt} \geq \dfrac{64xyzt}{(x+y+z+t)^3}$.
Thus, suffice to prove inequality
(**4**) $x+y+z+t+\dfrac{64xyzt}{(x+y+z+t)^3} \geq 5(xyzt)^{1/4}$.
Let $x+y+z+t=1$(due homogeneity of (**2**)) and let $u := (xyzt)^{1/4} \leq \dfrac{x+y+z+t}{4} = \dfrac{1}{4}$.
Then inequality (**4**) becomes $1+64u^4 \geq 5u$ and we have
$$1+64u^4-5u = (1-4u)(1-u-4u^2-16u^3) \geq$$
$$(1-4u)\left(1-\frac{1}{4}-4\cdot\frac{1}{16}-16\cdot\frac{1}{64}\right) = \frac{1-4u}{4} \geq 0.$$
Remark.
As above, denoting arithmetic, harmonic and geometric means of x,y,z,t by A,H,G, respectively we can rewrite inequality (**b**) as $4A+H \geq 5G$ and inequality (**3**) as $A^3H \geq G^4$.

★ **Generalization**:
Let $A := \dfrac{x_1+x_2+\ldots+x_n}{n}, H := \dfrac{x_1^{-1}+x_2^{-1}+\ldots+x_n^{-1}}{n}, G := (x_1x_2\ldots x_n)^{1/n}$,
$S := \dfrac{x_1x_2\ldots x_n}{n}\left(\dfrac{1}{x_1}+\dfrac{1}{x_2}+\ldots+\dfrac{1}{x_n}\right) = G^nH^{-1}$.
We will prove inequality which generalize inequalities (**a**) and (**b**) that is inequality
(**5**) $nA+H \geq (n+1)G,\ n \geq 3$.

Since by Maclaurin's inequality $A^{n-1} \geq S \iff A^{n-1} \geq G^n H^{-1} \iff H \geq \dfrac{G^n}{A^{n-1}}$
then suffice to prove inequality
(**6**) $nA + \dfrac{G^n}{A^{n-1}} \geq (n+1)G.$
We have (**6**)$\iff nA^n + G^n - (n+1)G \geq 0$ and since $nA^n + G^n - (n+1)GA^{n-1} =$
$nA^{n-1}(A-G) - G(A^{n-1} - G^{n-1}) = (A-G)\left(nA^{n-1} - G\sum_{k=1}^{n-1} A^{n-k}G^{k-1}\right) \geq$
$(A-G)\left(nA^{n-1} - A\sum_{k=1}^{n-1} A^{n-k}A^{k-1}\right) = (A-G)(nA^{n-1} - (n-1)A^{n-1}) = A^{n-1}(A-G) \geq 0$
inequality (**6**) is proved.